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# ***On the Functions Defined by Differential Equations, with an Extension of the Puiseux Polygon Construction to these Equations.***

BY HENRY B. FINE.

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In their memoir *Propriétés des fonctions définies par des équations différentielles* (Journal de l'Ecole Pol. Cah. 36) Briot and Bouquet present methods for obtaining developments for all ordinary solutions of a differential equation of the first order  $f(x, y, p) = 0$  which belong to the initial values  $x = y = 0$  of the variables, when it is known that  $p$  as well as  $y$  vanishes with  $x$ .

But as a general equation  $f(x, y, p) = 0$  which has no term independent of  $x, y$ , or  $p$  may very well have groups of terms of lower degree in respect to  $x$  than the remaining terms of the equation, for which  $p$  does not vanish with  $x$ , but remains finite and different from zero, or becomes infinite—and yet the corresponding  $y$  does vanish and is a solution of the equation,—it often becomes necessary at the very outset, and directly from the differential equation itself, to make a determination of all the possible groups of terms of lowest dimension, and this Briot and Bouquet give no means of doing.

In the first section of the present paper it is shown how this determination may be very simply accomplished by an extension of the polygon construction used by Puiseux in his study of algebraic functions.\* This done, the reduction of the cases where  $p$  does not vanish with  $x$  to the case so fully discussed by Briot and Bouquet is easy.

In the second section, after extending the polygon construction to the equation of the  $n^{\text{th}}$  order, and so determining for it also the terms of lowest dimension when the equation contains no term independent of  $x, y$ , or one of the differential coefficients  $y_1, y_2, \dots y_n$ ,—I give methods for obtaining the corresponding developments themselves. It is proven also that these developments are

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\* Journal de Math. pure et appliquées, I, 15.

actual as well as formal solutions of the equation, for it is shown that they converge for values of  $x$  which are greater than zero.

The method, of course, leads not only to all "monodrome" integrals which vanish with  $x$ , but to those also which belong to any initial values  $x^0, y^0, y_1^0, \dots y_{n-1}^0$  of the variables and the differential coefficients of lower orders, since the equation may in this case be readily transformed into one which has no absolute term.

### §1.

Let  $f(x, y, p) \equiv \Sigma A_i x^{\alpha_i} y^{\beta_i} p^{\gamma_i} = 0$

be any equation of the first order which has no term independent of  $x, y$ , or  $p$ .

It is proposed to make every determination of the terms of lowest degree in  $\Sigma A_i x^{\alpha_i} y^{\beta_i} p^{\gamma_i} = 0$  which is possible on the assumption that  $y$  vanishes with  $x$ .

In every case represent by  $\mu$  the degree of  $y$  in respect to  $x$ ; by the hypothesis that  $y$  vanishes with  $x$ ,  $\mu > 0$ .

Let  $A_1 x^{\alpha_1} y^{\beta_1} p^{\gamma_1}$  be one of the required terms of lowest degree; there must be at least one other term, say  $A_2 x^{\alpha_2} y^{\beta_2} p^{\gamma_2}$ , of the same degree, and a comparison of the two gives for the corresponding  $\mu$  the equation

$$\alpha_1 + \mu\beta_1 + (\mu - 1)\gamma_1 = \alpha_2 + \mu\beta_2 + (\mu - 1)\gamma_2,$$

whence

$$\mu = -\frac{\alpha_1 - \gamma_1 - (\alpha_2 - \gamma_2)}{\beta_1 + \gamma_1 - (\beta_2 + \gamma_2)}.$$

Take two rectangular axes  $\eta, \xi$ , and construct a point  $\xi_i = \alpha_i - \gamma_i$ ,  $\eta_i = \beta_i + \gamma_i$  to correspond to each term  $A_i x^{\alpha_i} y^{\beta_i} p^{\gamma_i}$ . Then the line joining  $\xi_1 \eta_1$  and  $\xi_2 \eta_2$ , viz. the line whose equation is

$$\xi - \xi_1 = \frac{\alpha_1 - \gamma_1 - (\alpha_2 - \gamma_2)}{\beta_1 + \gamma_1 - (\beta_2 + \gamma_2)} (\eta - \eta_1) = -\mu (\eta - \eta_1)$$

makes with the  $\eta$ -axis an angle of which the tangent is  $-\mu$ , and cuts off on the  $\xi$ -axis an intercept

$$\xi_1 + \mu\eta_1 = \alpha_1 - \gamma_1 + \mu(\beta_1 + \gamma_1),$$

which is equal to the common degree of the two terms  $A_1 x^{\alpha_1} y^{\beta_1} p^{\gamma_1}$ ,  $A_2 x^{\alpha_2} y^{\beta_2} p^{\gamma_2}$ .

Since  $\mu > 0$  the line makes an oblique angle with the  $\eta$ -axis.

Furthermore, a parallel to this line through any of the other points  $\xi_i, \eta_i$  cuts off on the  $\xi$ -axis an intercept  $\alpha_i - \gamma_i + \mu(\beta_i + \gamma_i)$ , equal to the degree of the corresponding term. If, therefore,  $A_1 x^{\alpha_1} y^{\beta_1} p^{\gamma_1}$ ,  $A_2 x^{\alpha_2} y^{\beta_2} p^{\gamma_2}$  be, as was supposed,

terms of lowest degree, all the other points must lie to the same side of the line  $\xi_1\eta_1 - \xi_2\eta_2$  as the origin when its intercept is negative, to the opposite side when its intercept is positive.

Hence to get every admissible group of lowest terms in the equation  $\Sigma A_i x^{\alpha_i} y^{\beta_i} p^{\gamma_i} = 0$ —that is, every group of terms for which the corresponding  $\mu$  is positive and such as to make the terms of the group of lower degree than the remaining terms of the equation—move up a parallel to the  $\eta$ -axis from a position below any of the  $\xi_i\eta_i$  points until it meets one of these points or a group of them; next turn it (clockwise, since  $\mu > 0$ ) about the point of this group which is nearest the  $\xi$ -axis until it meets a second point or group of points; again turn it about the point of this second group which is nearest the  $\xi$ -axis, and so on until further turning would bring it past the position of parallelism with the  $\xi$ -axis.

To each side of the polygon thus constructed—except that parallel to the  $\eta$ -axis, should it occur—correspond one or more developments of  $y$  in increasing powers of  $x$ , each beginning with the term  $x^\mu$  and—save in exceptional cases—satisfying the equation  $f(x, y, p) = 0$ .

A side parallel to the  $\eta$ -axis is to be rejected, since for it  $\mu = 0$ , or the corresponding  $y$  does not vanish with  $x$ .

The construction can make a parallel to the  $\xi$ -axis a polygon side only in case there be no mere  $x$  term in the equation—when  $y = 0$  is a solution. Not all equations, however, of which  $y = 0$  is a solution have this line for a polygon side; the equation  $p + py + y^3 = 0$  does not, for instance.

The developments corresponding to any polygon side for which  $\mu > 1$  can be obtained immediately by the Briot-Bouquet methods already referred to, since here  $p$  as well as  $y$  vanishes with  $x$ .

Those corresponding to a side for which  $\mu < 1$  are to be obtained by the same methods after an interchange has been made of the dependent and independent variables.

If for any side  $\mu = 1$ , make in  $f(x, y, p) = 0$  the substitutions

$$y = vx, p = v + \frac{dv}{dx} x,$$

where  $v$  takes a finite value  $v_0$  different from zero when  $x = 0$ .

The values of  $v_0$  belonging to the various developments are given by the group of terms of lowest order in  $f(x, y, p) = 0$ , to which the side  $\mu = 1$  belongs, and by the substitution  $y = (v_0 + y')x$ ,  $f = 0$  is again reduced to the Briot-Bouquet form.

The various developments in all these cases may also be obtained by making in  $f=0$  the substitutions  $x=t^s$ ,  $y=t^r v$ ,  $p=\frac{t^{r-s}}{s}\left(rv+t\frac{dv}{dt}\right)$ ,—where  $\frac{r}{s}=\mu$ ,—and determining the corresponding  $v$ 's from the transformed equation. But this method is less direct and elegant than that of Briot-Bouquet.

## §2.

It is obvious that the polygon construction is equally applicable to equations of higher orders.

The equation being  $\Sigma x^{\alpha_i} y^{\beta_i} y_1^{\gamma_i} \dots y_n^{\nu_i} = 0$ , supposed to have no term independent of  $x, y, \dots$  or  $y_n$ , take as before axes of  $\xi$  and  $\eta$ , and following the method already explained for the equation of the first order, construct points

$$\xi_i = \alpha_i - \gamma_i - 2\delta_i - \dots - n\nu_i, \quad \eta_i = \beta_i + \gamma_i + \delta_i + \dots + \nu_i$$

to correspond to the various terms  $A_i x^{\alpha_i} y^{\beta_i} \dots y_n^{\nu_i}$  of the equation. The polygon may then be constructed exactly as for the equation of the first order, and similar inferences drawn with reference to the terms of lowest order in the equation.

In the case of the equation of the  $n^{\text{th}}$  order it is necessary, when deriving the developments corresponding to the various polygon sides, to distinguish between sides whose  $\mu$  is  $>n$  and those whose  $\mu$  is  $=$  or  $<n$ . For the first  $y_n$  as well as  $y$  and all its lower differential coefficients vanish with  $x$ ; for the second,  $y_n$  takes a finite value different from zero or becomes infinite when  $x$  vanishes.

It is always possible, however, by a simple transformation, to reduce the second case to the first. For if  $\mu=n$ , set

$$y = vx^n, \quad y_1 = nvx^{n-1} + x^n \frac{dv}{dx}, \text{ etc.}$$

in the group of lowest terms under consideration, remove the common factor  $x^n$  and set the group equal to zero. The roots  $v_0$  of the equation thus constructed are the initial values of  $v$  in the set of developments sought for; and the substitution  $y = x^n(v_0 + y')$ —where  $y'$  vanishes with  $x$ —effects the required transformation.

If, on the other hand,  $\mu < n$  for any side or series of sides, select the side whose  $\mu$  is least, find  $m$ , the first integer greater than the quotient of  $n$  by this  $\mu$ , and make the substitution  $x = x'^m$ . Then for all the groups of lowest terms in the transformed equation  $\mu'$ , the degree of  $y$  in respect to  $x'$ , is greater than  $n$ .

It only remains therefore to indicate a general method for getting the various developments corresponding to a polygon side for which  $\mu > n$ .

This done, it may be added, we are in position to derive all solutions of the equation

$$f \equiv f_0 y_n^m + f_1 y_n^{m-1} + \dots + f_m = 0$$

which belong to *any* given initial values  $x_0, y_0, \dots, y_{n-1}^0$  of  $x, y$  and the lower differential coefficients of  $y$ .  $f_0, f_1, \dots, f_m$  are supposed to be holomorphic functions of  $x, y, \dots, y_{n-1}$  for a common region of convergence, and  $x_0, y_0, \dots, y_{n-1}^0$  to lie within this region.

For if  $f_m$  does not vanish for  $x = x_0, y = y_0$ , etc., the corresponding initial values  $y_n^0$  of  $y_n$  may, unless  $f_0$  vanishes, be immediately obtained from  $f = 0$ , which is algebraic in  $y_n$ ; and the substitutions

$$\begin{aligned} x &= x_0 + x', \\ \frac{d^\kappa y}{dx^\kappa} &= y_\kappa^0 + y_{\kappa+1}^0 x' + y_{\kappa+2}^0 \frac{x'^2}{2!} + \dots + y_n^0 \frac{x'^{n-\kappa}}{(n-\kappa)!} + y'_\kappa, \quad (\kappa = 0, 1, \dots, n) \end{aligned}$$

transform  $f = 0$  into an equation in  $x', y', y'_1, \dots, y'_n$  which has no term independent of one or other of these variables, and for which also  $y'_n$ , as well as the lower differential coefficients of  $y'$ , vanishes with  $x'$ .

If, on the other hand,  $f_0$  vanishes, apply the polygon construction already described to  $f = 0$ , regarded as an equation in  $x_0 + x', y_0 + y', y_1^0 + y'_1, \dots, y_{n-1}^0 + y'_{n-1}$ , and  $\frac{1}{y_n}$  (it has no term which does not involve one of these variables), and the various values of  $\mu$  having been obtained, transform as above, by the substitution  $x = x'^m$ , into an equation whose  $n^{\text{th}}$  differential coefficient vanishes with  $x'$ .

To get the developments corresponding to a polygon side for which  $\mu > n$ , that is in the case where all the differential coefficients  $y_1, y_2, \dots, y_n$  as well as  $y$  vanish with  $x$ , make in the equation the substitutions

$$\left. \begin{aligned} y_{n-1} &= y_n v_1 x, \\ y_{n-2} &= y_n v_1 v_2 x^2, \\ &\dots \dots \dots \\ y &= y_n v_1 v_2 \dots v_n x, \end{aligned} \right\} \quad (1)$$

where  $v_1, v_2, \dots, v_n$  are functions of  $x$  which take finite values  $v_1^0, v_2^0, \dots, v_n^0$  different from zero when  $x = 0$ .

In the resulting equation, freed from any factor  $y_n^\alpha x^\beta v_1^\gamma \dots v_{n-1}^\nu$  which may be common to all the terms, determine by the ordinary Puiseux method or any

other method applicable to algebraic functions of a single variable, the various groups of terms of lowest order in  $y_n$  and  $x$ : if indeed this determination has not already been made directly from the equation.

Suppose that for any particular group of terms of lowest order the degree of  $y_n$  in respect to  $x$  is  $\frac{r}{s}$ : to obtain the corresponding developments make then the further substitutions

$$x = x'^s, y_n = Vx'^r, \quad (2)$$

where again  $V$  takes an initial value  $V_0$  different from 0 and  $\infty$  when  $x'$  vanishes.

Since  $y_n = V_0 x'^{\frac{r}{s}}$  approximately,  $y_{n-1} = \frac{s}{r+s} V_0 x'^{\frac{r+s}{s}} + \dots$ ; but

$$y_{n-1} = y_n v_1 x = V_0 v_1^0 x'^{\frac{r+s}{s}} + \dots;$$

therefore, by a comparison of the two values of  $y_{n-1}$ ,  $v_1^0 = \frac{s}{r+s}$ . In like manner  $v_2^0 = \frac{s}{r+2s}, \dots, v_n^0 = \frac{s}{r+ns}$ .

In the  $V$  equation therefore, make the substitutions

$$v_1 = \frac{s}{r+s} + v'_1, \quad v_2 = \frac{s}{r+2s} + v'_2, \dots, v_n = \frac{s}{r+ns} + v'_n \quad (3)$$

when  $V$  will be given as a function of  $x', v'_1, \dots, v'_n$ , developable in integral powers of these variables.

The various developments for  $V$  having been obtained, the equations for the corresponding sets of values of  $v'_1, v'_2, \dots, v'_n$  are readily constructed by aid of the equations (1), (2), (3). For evidently on introducing the substitutions (2) in equations (1) we have:

$$\begin{aligned} Vx'^r &= \frac{d}{dx} (Vv_1 x'^{r+s}), \\ Vv_1 x'^{r+s} &= \frac{d}{dx} (Vv_1 v_2 x'^{r+2s}), \\ &\dots \dots \dots \\ Vv_1 v_2 \dots v_{n-1} x'^{r+(n-1)s} &= \frac{d}{dx} (Vv_1 v_2 \dots v_n x'^{r+ns}), \end{aligned}$$



a set of linear equations in  $\left(\frac{dv'_1}{dx'}\right)_0$ ,  $\left(\frac{dv'_2}{dx'}\right)_0$ , etc., which give finite determinate values for these quantities except when the determinant of their coefficients vanishes.

A second differentiation gives the equations

$$\begin{aligned} 2 \left(\frac{d^2v'_1}{dx'^2}\right)_0 &= \left(\phi_1 + \frac{\partial f_1}{\partial v'_1} \frac{d^2v'_1}{dx'^2} + \frac{\partial f_1}{\partial v'_2} \frac{d^2v'_2}{dx'^2} + \dots \frac{\partial f_1}{\partial v'_n} \frac{d^2v'_n}{dx'^2}\right)_0, \\ 2 \left(\frac{d^2v'_i}{dx'^2}\right)_0 &= \left(\phi_i + \frac{\partial f_i}{\partial v'_{i-1}} \frac{d^2v'_{i-1}}{dx'^2} + \frac{\partial f_i}{\partial v'_2} \frac{d^2v'_2}{dx'^2}\right)_0, \quad i = 2, 3, \dots, n \end{aligned}$$

where  $\phi_1^0$ ,  $\phi_i^0$  are functions of  $\left(\frac{dv'_1}{dx'}\right)_0$ ,  $\left(\frac{dv'_2}{dx'}\right)_0$ , etc., and of second differential coefficients of the  $f$ 's; and from these equations the values of  $\left(\frac{d^2v'_1}{dx'^2}\right)_0$ ,  $\left(\frac{d^2v'_2}{dx'^2}\right)_0$ , etc., may be reckoned.

Further differentiations give in like manner the higher differential coefficients. It is only necessary, therefore, for the construction of series

$$v'_i = \left(\frac{dv'_i}{dx'}\right)_0 x' + \frac{1}{2!} \left(\frac{d^2v'_i}{dx'^2}\right)_0 x'^2 + \frac{1}{3!} \left(\frac{d^3v'_i}{dx'^3}\right)_0 x'^3 + \dots, \quad i = 1, 2, \dots, n \quad (6)$$

which *formally* satisfy the equations, that the determinant of the coefficients in each of these sets of equations shall be different from 0, or that

$$\begin{vmatrix} \frac{\partial f_1}{\partial v'_1} - \kappa, & \frac{\partial f_1}{\partial v'_2}, & \frac{\partial f_1}{\partial v'_3}, & \dots, & \frac{\partial f_1}{\partial v'_n} \\ \frac{\partial f_2}{\partial v'_1}, & \frac{\partial f_2}{\partial v'_2} - \kappa, & 0, & \dots, & 0 \\ 0 & \frac{\partial f_3}{\partial v'_2}, & \frac{\partial f_3}{\partial v'_3} - \kappa, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{\partial f_n}{\partial v'_{n-1}}, & \frac{\partial f_n}{\partial v'_n} - \kappa \end{vmatrix}_{x=0},$$

or

$$\left. \begin{aligned} & V_0(r+s+\kappa)(r+2s+\kappa) \dots (r+ns+\kappa) \\ & + \frac{\partial V_0}{\partial v'_1} \frac{\kappa s}{r+s} (r+2s+\kappa) \dots (r+ns+\kappa) \\ & + \frac{\partial V_0}{\partial v'_2} \frac{\kappa s(r+s)}{r+2s} (r+3s+\kappa) \dots (r+ns+\kappa) \\ & + \dots \\ & + \frac{\partial V_0}{\partial v'_n} \frac{\kappa s(r+s)(r+2s) \dots r+(n-1)s}{r+ns} \end{aligned} \right\} \quad (7)$$

shall vanish for no positive integral value of  $\kappa$ .



As above, it will be assumed that the determinant

$$\Delta(\kappa) \equiv \begin{vmatrix} \frac{\partial f_1}{\partial v_1} - \kappa, & \frac{\partial f_1}{\partial v_2}, & \frac{\partial f_1}{\partial v_3}, & \dots & \frac{\partial f_1}{\partial v_n} \\ \frac{\partial f_2}{\partial v_1}, & \frac{\partial f_2}{\partial v_2} - \kappa, & \frac{\partial f_2}{\partial v_3}, & \dots & \frac{\partial f_2}{\partial v_n} \\ \frac{\partial f_3}{\partial v_1}, & \frac{\partial f_3}{\partial v_2}, & \frac{\partial f_3}{\partial v_3} - \kappa, & \dots & \frac{\partial f_3}{\partial v_n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial v_1}, & \frac{\partial f_n}{\partial v_2}, & \frac{\partial f_n}{\partial v_3}, & \dots & \frac{\partial f_n}{\partial v_n} - \kappa \end{vmatrix}_{x=0} \quad (10)$$

vanishes for no positive integral value of  $\kappa$ .

Solving the equations (9) we have

$$\left(\frac{d^\kappa v_i}{dx^\kappa}\right)_0 = \phi_{\kappa 1}^0 \frac{\Delta_{1i}(\kappa)}{\Delta(\kappa)} + \phi_{\kappa 2}^0 \frac{\Delta_{2i}(\kappa)}{\Delta(\kappa)} + \dots + \phi_{\kappa n}^0 \frac{\Delta_{ni}(\kappa)}{\Delta(\kappa)}; \quad i = 1, 2, \dots, n \quad (11)$$

where  $\Delta_{ji}(\kappa)$  is the minor of the element in the  $j^{\text{th}}$  row and  $i^{\text{th}}$  column of  $\Delta(\kappa)$ .

Arranged with reference to the powers of  $\kappa$ , the coefficients  $\frac{\Delta_{ji}(\kappa)}{\Delta(\kappa)}$  have the form

$$\frac{\alpha_{ji}\kappa^{n-1} + \psi_{ji,1}\kappa^{n-2} + \psi_{ji,2}\kappa^{n-3} + \dots + \psi_{ji,n-1}}{\kappa^n + \chi_1\kappa^{n-1} + \chi_2\kappa^{n-2} + \dots + \chi_n}$$

where  $\alpha_{ji} = 0$  when  $j \geq i$ ;  $= 1$  when  $j = i$ .

As the coefficients in this fraction are independent of  $\kappa$  and known, and as furthermore its denominator vanishes for no value that  $\kappa$  can take, and is of higher degree in  $\kappa$  than its numerator, it must reach a greatest value, and that  $< \infty$ , for some finite value of  $\kappa$ . Let this greatest value be  $C_{ji}$ ; and let  $C$  be the modulus of the greatest of the quantities  $C_{ji}$  ( $j, i = 1, 2, \dots, n$ ). Then

$$\left(\frac{d^\kappa v_i}{dx^\kappa}\right)_0 \leq C(\phi_{\kappa,1}^0 + \phi_{\kappa,2}^0 + \dots + \phi_{\kappa,n}^0). \quad (12)$$

Let now  $r$  be  $\leq$  the least of the radii  $r_1, r_2, \dots, r_n$ , and let  $M$  be the modulus of the greatest value which any of the functions  $f_1, f_2, \dots, f_n$  takes in the circle of radius  $\rho$  about  $x = 0$  and the circles of radius  $r$  about  $v_1 = 0, v_2 = 0$ , etc.

Let also  $a$  be the modulus of the greatest of the differential coefficients of the first order  $\left(\frac{dv_i}{dx}\right)_0$ , and set  $a' = a - \frac{nCM}{\rho}$ .

Consider, then, the equation, algebraic in  $u$  and  $x$ ,

$$u = a'x + nC\Phi,$$

where

$$\Phi = M \left( -n \frac{u}{r} - 1 + \frac{1}{\left(1 - \frac{u}{r}\right)^n \left(1 - \frac{x}{\rho}\right)} \right). \quad (13)$$

It may readily be shown that  $\left(\frac{d^\kappa u}{dx^\kappa}\right)_0 > \left(\frac{d^\kappa v_i}{dx^\kappa}\right)_0$  for  $\kappa > 1$ . For  $\left(\frac{\partial \Phi}{\partial u}\right)_0 = 0$ , so that  $\left(\frac{d^\kappa \Phi}{dx^\kappa}\right)_0$  involves the differential coefficients  $\left(\frac{du}{dx}\right)_0$ ,  $\left(\frac{d^2 u}{dx^2}\right)_0$  etc. of orders less than  $\kappa$  only and the equation

$$\left(\frac{d^\kappa u}{dx^\kappa}\right)_0 = nC \left(\frac{d^\kappa \Phi}{dx^\kappa}\right)_0$$

gives  $\left(\frac{d^\kappa u}{dx^\kappa}\right)_0$  explicitly in terms of the partial differential coefficients of  $\Phi$  in respect to  $u$  and  $x$  and of the differential coefficients of lower orders of  $u$  in respect to  $x$ .

Now  $\left(\frac{d^\kappa \Phi}{dx^\kappa}\right)_0$  is greater than any of the functions  $\phi_{\kappa i}^0$  in equation (12). For  $\phi_{\kappa i} + \frac{\partial f_i}{\partial v_1} \frac{d^\kappa v_1}{dx^\kappa} + \frac{\partial f_i}{\partial v_2} \frac{d^\kappa v_2}{dx^\kappa} + \dots + \frac{\partial f_i}{\partial v_n} \frac{d^\kappa v_n}{dx^\kappa}$  is the result of operating on  $f_i$   $\kappa$  times with  $\frac{\partial}{\partial x} + \frac{\partial}{\partial v_1} \frac{dv_1}{dx} + \dots + \frac{\partial}{\partial v_n} \frac{dv_n}{dx}$ , while  $\frac{d^\kappa \Phi}{dx^\kappa}$  is the result of operating on  $\Phi$   $\kappa$  times with  $\frac{\partial}{\partial x} + n \frac{\partial}{\partial u} \frac{du}{dx}$ .

To every differential coefficient  $\frac{\partial^{n+n'+n''+\dots} f_i}{\partial x^n \partial v_1^{n'} \partial v_2^{n''} \dots}$  in  $\phi_{\kappa i}$  therefore corresponds the differential coefficient  $\frac{\partial^{n+n'+n''+\dots} \Phi}{\partial x^n \partial u^{n'+n''+\dots}}$  in  $\frac{d^\kappa \Phi}{dx^\kappa}$ , involved with the same numerical coefficient; and to every  $\frac{d^j v_i}{dx^j}$  in  $\phi_{\kappa i}$  ( $j < \kappa$ ), corresponds  $\frac{d^j u}{dx^j}$  in  $\frac{d^\kappa \Phi}{dx^\kappa}$ .

But, as is known,\*

$$\text{mod} \left( \frac{\partial^{n+n'+n''+\dots} f_i}{\partial x^n \partial v_1^{n'} \partial v_2^{n''} \dots} \right)_0 < n! \, n'! \, n''! \, \dots \, \frac{M}{\rho^n r_1^{n'} r_2^{n''} \dots},$$

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\* Vid. Briot and Bouquet, Fonc. ellip. p. 326.

while 
$$\left(\frac{\partial^{n+n'+n''+\dots}\Phi}{\partial x^n \partial u^{n'+n''+\dots}}\right)_0 = n! n'! n''! \dots \frac{M}{\rho^n r_1^{n'} r_2^{n''} \dots} :$$

so that  $\text{mod} \left(\frac{\partial^{n+n'+n''+\dots}f_i}{\partial x^n \partial v_1^{n'} \partial v_2^{n''} \dots}\right)_0 < \left(\frac{\partial^{n+n'+n''+\dots}\Phi}{\partial x^n \partial u^{n'+n''+\dots}}\right)_0$ .

If, therefore,  $\left(\frac{d^j v_i}{dx^j}\right)_0 \leq \left(\frac{d^j u}{dx^j}\right)_0$  for  $j < \kappa$ , it is clear that the individual terms of  $\left(\frac{d^\kappa \Phi}{dx^\kappa}\right)_0$  are greater than the corresponding terms of  $\Phi_{\kappa i}^0$ , and since they are all positive, that  $\left(\frac{d^\kappa \Phi}{dx^\kappa}\right)_0 > \Phi_{\kappa i}^0$ , and hence that  $\left(\frac{d^\kappa u}{dx^\kappa}\right)_0$ , which is equal to  $nC \frac{d^\kappa \Phi}{dx^\kappa}$ , is greater than  $C(\Phi_{\kappa 1}^0 + \Phi_{\kappa 2}^0 + \dots \Phi_{\kappa n}^0)$ ; in other words (vid. 12), that  $\left(\frac{d^\kappa u}{dx^\kappa}\right)_0 > \left(\frac{d^\kappa v_i}{dx^\kappa}\right)_0$ .

But by hypothesis  $\left(\frac{du}{dx}\right)_0 \geq \left(\frac{dv_i}{dx}\right)_0$ . It follows at once that  $\left(\frac{d^2 u}{dx^2}\right)_0 > \left(\frac{d^2 v_i}{dx^2}\right)_0$ , therefore that  $\left(\frac{d^3 u}{dx^3}\right)_0 > \left(\frac{d^3 v_i}{dx^3}\right)_0, \dots$ , etc.

But  $u$  is defined as a holomorphic function of  $x$  for the neighborhood of  $x = 0$  by the algebraic equation (13), or

$$\psi(u_1 x) \equiv u \left(1 - \frac{u}{r}\right)^n \left(1 - \frac{x}{\rho}\right) \left(1 + \frac{bn}{r}\right) - \left(1 - \frac{u}{r}\right)^n (a'x - b) \left(1 - \frac{x}{\rho}\right) - b = 0; \quad (b = nCM),$$

namely within a circle about  $x = 0$  whose radius is the distance to the nearest branch point or singular point of  $\psi = 0$ . This distance is greater than zero, being the modulus of the least root of the discriminant of  $\psi = 0$  in respect to  $u$ , that is, of the equation

$$\left(1 - \frac{x}{\rho}\right) \left(1 + \frac{b - a'x}{r + bn}\right)^{n+1} = \frac{1}{1 + \frac{r}{bn}} \frac{(n+1)^{n+1}}{n^{n+1}}.$$

Within this circle, therefore, the series

$$u = \left(\frac{du}{dx}\right)_0 x + \left(\frac{d^2 u}{dx^2}\right)_0 \frac{x^2}{2!} + \left(\frac{d^3 u}{dx^3}\right)_0 \frac{x^3}{3!} + \dots$$

is convergent; and since its coefficients are greater than the corresponding coefficients of any of the series

$$v_i = \left(\frac{dv_i}{dx}\right)_0 x + \left(\frac{d^2 v_i}{dx^2}\right)_0 \frac{x^2}{2!} + \left(\frac{d^3 v_i}{dx^3}\right)_0 \frac{x^3}{3!} + \dots;$$

these series also converge for the same region, as was to be demonstrated.